Chaotic behavior of disordered nonlinear systems

Haris Skokos

Department of Mathematics and Applied Mathematics,
University of Cape Town

Cape Town, South Africa

E-mail: haris.skokos@uct.ac.za
URL: http://math_research.uct.ac.za/~hskokos/

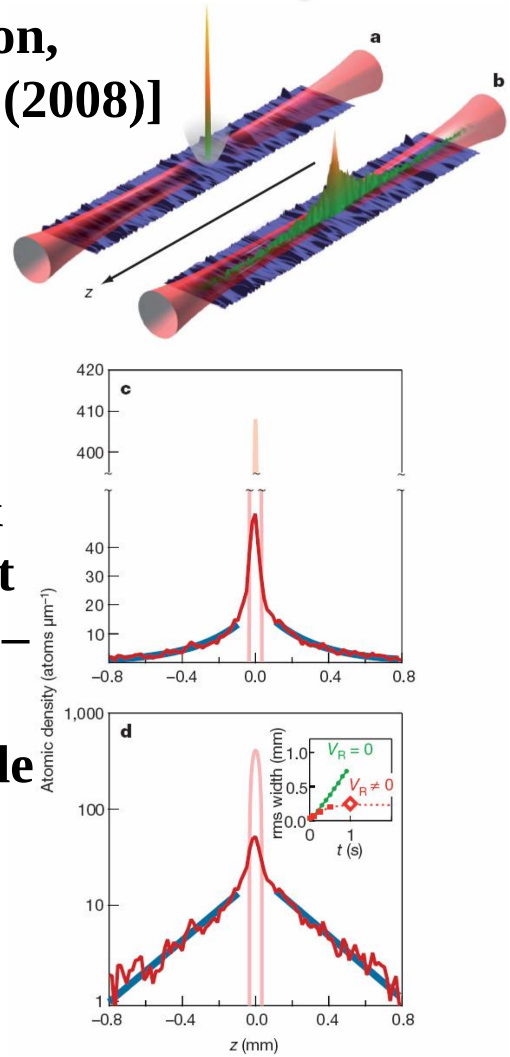
Work in collaboration with
Sergej Flach, Joshua Bodyfelt, Ioannis Gkolias,
Dima Krimer, Stavros Komineas, Tanya Laptyeva, Bob Senyange

Outline

- **Disordered 1D lattices:**
  - ✓ **The quartic Klein-Gordon (KG) model**
  - ✓ **The disordered nonlinear Schrödinger equation (DNLS)**
  - ✓ **Different dynamical behaviors**
- **Chaotic behavior of the KG model**
  - ✓ **Lyapunov exponents**
  - ✓ **Deviation Vector Distributions**
- **Integration techniques (Symplectic integrators and Tangent Map method)**
- **Summary**

Interplay of disorder and nonlinearity

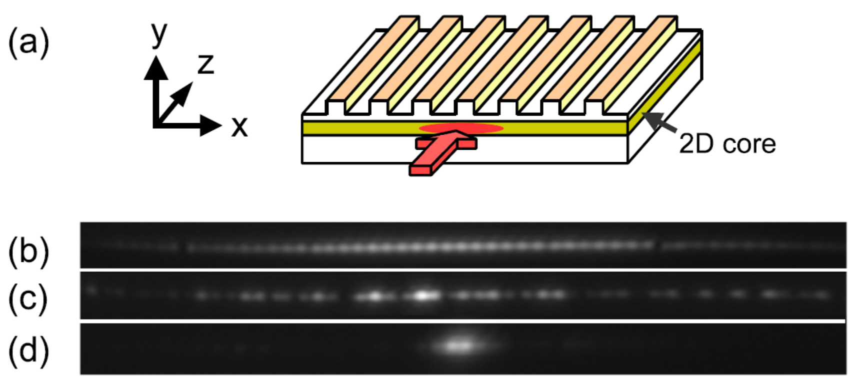
Waves in disordered media – Anderson localization [Anderson, Phys. Rev. (1958)]. Experiments on BEC [Billy et al., Nature (2008)]



Waves in nonlinear disordered media – localization or delocalization?

Theoretical and/or numerical studies [Shepelyansky, PRL (1993) – Molina, Phys. Rev. B (1998) – Pikovsky & Shepelyansky, PRL (2008) – Kopidakis et al., PRL (2008) – Flach et al., PRL (2009) – S. et al., PRE (2009) – Mulansky & Pikovsky, EPL (2010) – S. & Flach, PRE (2010) – Laptyeva et al., EPL (2010) – Mulansky et al., PRE & J.Stat.Phys. (2011) – Bodyfelt et al., PRE (2011) – Bodyfelt et al., IJBC (2011)]

Experiments: propagation of light in disordered 1d waveguide lattices [Lahini et al., PRL (2008)]



The Klein – Gordon (KG) model

$$H_K = \sum_{l=1}^{N} \frac{p_l^2}{2} + \frac{\tilde{\varepsilon}_l}{2} u_l^2 + \frac{1}{4} u_l^4 + \frac{1}{2W}\left(u_{l+1} - u_l\right)^2$$

with fixed boundary conditions $u_0=p_0=u_{N+1}=p_{N+1}=0$. Typically N=1000.

Parameters: W and the total energy E. $\tilde{\varepsilon}_l$ chosen uniformly from $\left[\frac{1}{2},\frac{3}{2}\right]$.

Linear case (neglecting the term $u_l^4/4$)

Ansatz: $u_l=A_l \exp(i\omega t)$. Normal modes (NMs) $A_{\nu,l}$ - Eigenvalue problem:

$$\lambda A_l = \varepsilon_l A_l - (A_{l+1} + A_{l-1}) \text{ with } \lambda = W\omega^2 - W - 2, \quad \varepsilon_l = W(\tilde{\varepsilon}_l - 1)$$

The discrete nonlinear Schrödinger (DNLS) equation

We also consider the system:

$$H_D = \sum_{l=1}^{N} \varepsilon_l \left|\psi_l\right|^2 + \frac{\beta}{2}\left|\psi_l\right|^4 - \left(\psi_{l+1}\psi_l^* + \psi_{l+1}^*\psi_l\right)$$

where ε_l chosen uniformly from $\left[-\frac{W}{2},\frac{W}{2}\right]$ and β is the nonlinear parameter.

Conserved quantities: The energy and the norm $S = \sum_l \left|\psi_l\right|^2$ of the wave packet.

Distribution characterization

We consider normalized energy distributions in normal mode (NM) space

$$z_\nu \equiv \frac{E_\nu}{\sum_m E_m} \quad \text{with} \quad E_\nu = \frac{1}{2}\left(\dot{A}_\nu^2 + \omega_\nu^2 A_\nu^2\right)$$

, where A_ν is the amplitude of the νth NM (KG) or norm distributions (DNLS).

Second moment: $$m_2 = \sum_{\nu=1}^{N} \left(\nu - \bar{\nu}\right)^2 z_\nu \quad \text{with} \quad \bar{\nu} = \sum_{\nu=1}^{N} \nu z_\nu$$

Participation number: $$P = \frac{1}{\sum_{\nu=1}^{N} z_\nu^2}$$

measures the number of stronger excited modes in z_ν.

Single mode $P=1$. Equipartition of energy $P=N$.

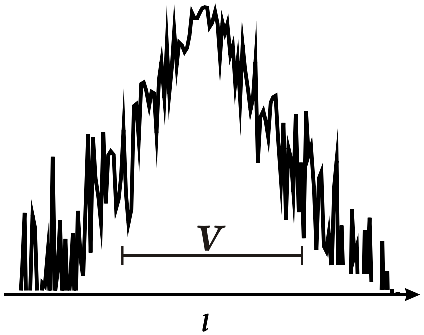
Scales

Linear case: $\omega_\nu^2 \in \left[\frac{1}{2}, \frac{3}{2} + \frac{4}{W}\right]$**, width of the squared frequency spectrum:**

$$\Delta_K = 1 + \frac{4}{W}$$

$$(\Delta_D = W + 4)$$

Localization volume of an eigenstate: $V \sim \frac{1}{\sum_{l=1}^{N} A_{\nu,l}^4}$



Average spacing of squared eigenfrequencies of NMs within the range of a localization volume: $d_K \approx \frac{\Delta_K}{V}$

Nonlinearity induced squared frequency shift of a single site oscillator

$$\delta_l = \frac{3E_l}{2\tilde{\varepsilon}_l} \propto E \qquad (\delta_l = \beta |\psi_l|^2)$$

The relation of the two scales $d_K \le \Delta_K$ with the nonlinear frequency shift δ_l determines the packet evolution.

Different Dynamical Regimes

Three expected evolution regimes [Flach, Chem. Phys (2010) - S. & Flach, PRE (2010) - Laptyeva et al., EPL (2010) - Bodyfelt et al., PRE (2011)]

Δ: width of the frequency spectrum, d: average spacing of interacting modes,

δ: nonlinear frequency shift.

Weak Chaos Regime: $\delta<d$, $m_2 \sim t^{1/3}$

Frequency shift is less than the average spacing of interacting modes. NMs are weakly interacting with each other. [Molina, PRB (1998) – Pikovsky, & Shepelyansky, PRL (2008)].

Intermediate Strong Chaos Regime: $d<\delta<\Delta$, $m_2 \sim t^{1/2} \longrightarrow m_2 \sim t^{1/3}$

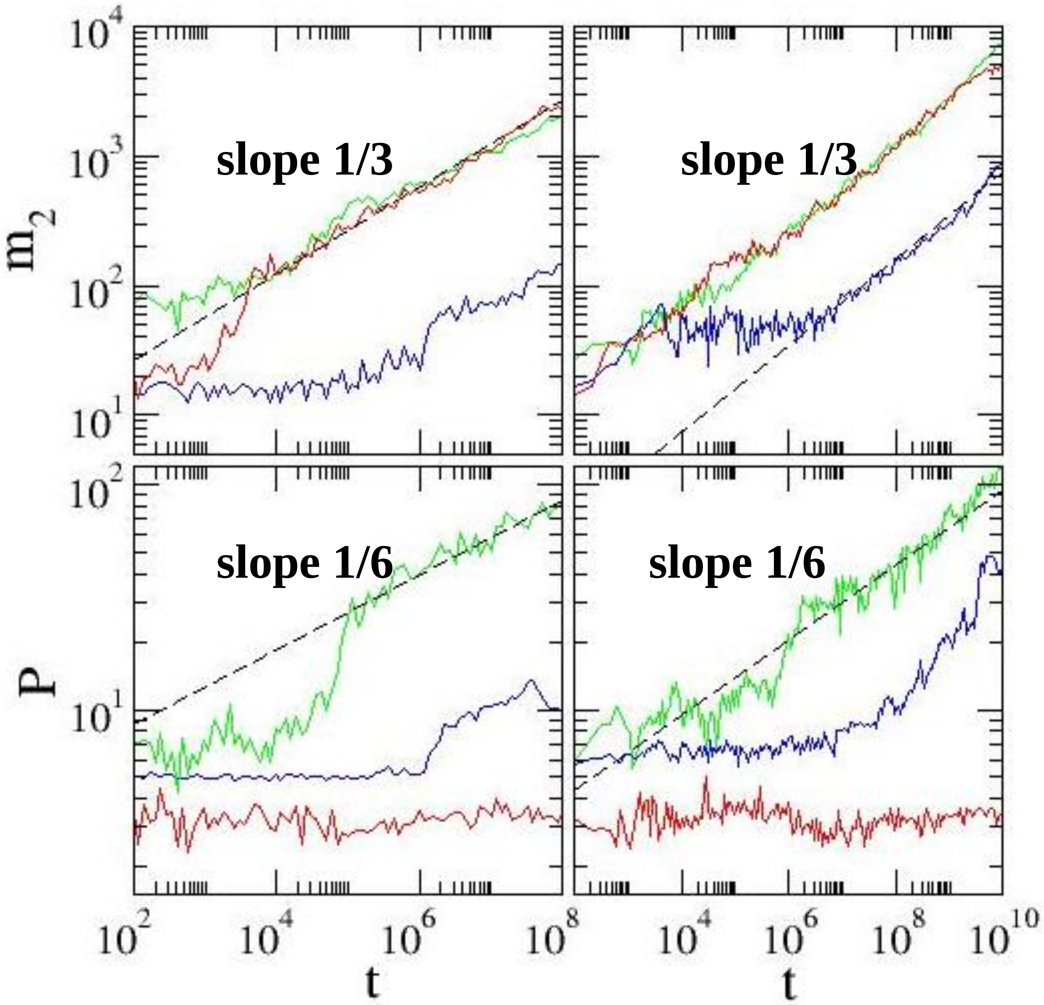
Almost all NMs in the packet are resonantly interacting. Wave packets initially spread faster and eventually enter the weak chaos regime.

Selftrapping Regime: $\delta>\Delta$

Frequency shift exceeds the spectrum width. Frequencies of excited NMs are tuned out of resonances with the nonexcited ones, leading to selftrapping, while a small part of the wave packet subdiffuses [Kopidakis et al., PRL (2008)].

Single site excitations

DNLS W=4, β= 0.1, 1, 4.5 **KG** W = 4, E = 0.05, 0.4, 1.5



No strong chaos regime

In weak chaos regime we averaged the measured exponent α ($m_2 \sim t^{\alpha}$) over 20 realizations:

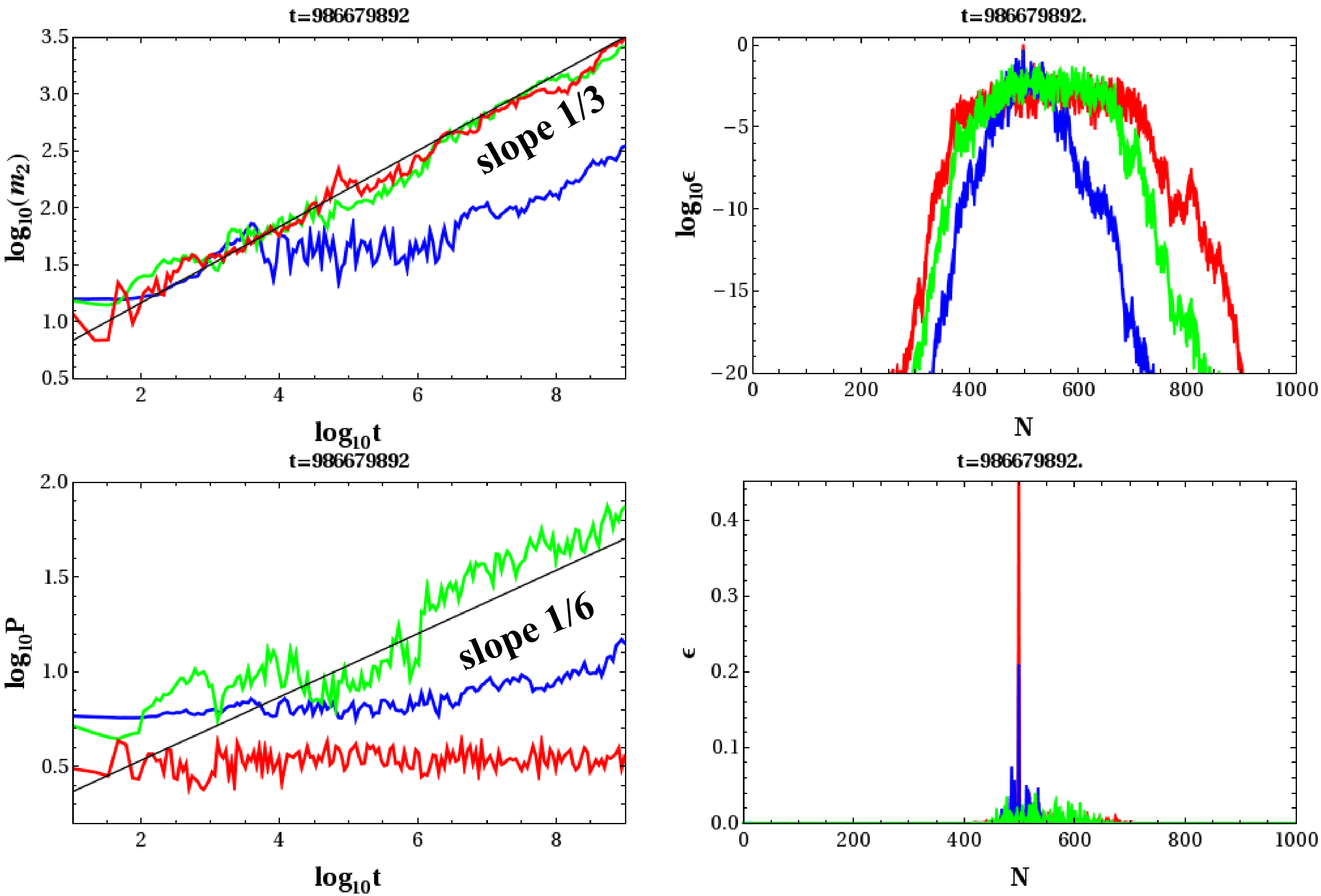
α=0.33±0.05 (KG)

α=0.33±0.02 (DLNS)

Flach et al., PRL (2009)

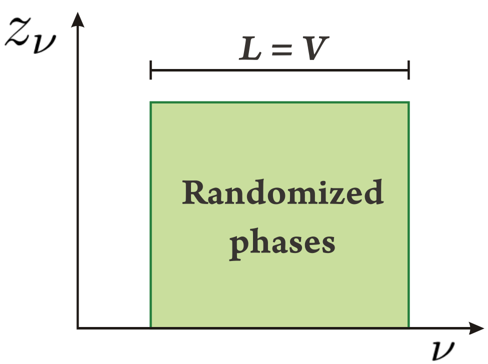
S. et al., PRE (2009)

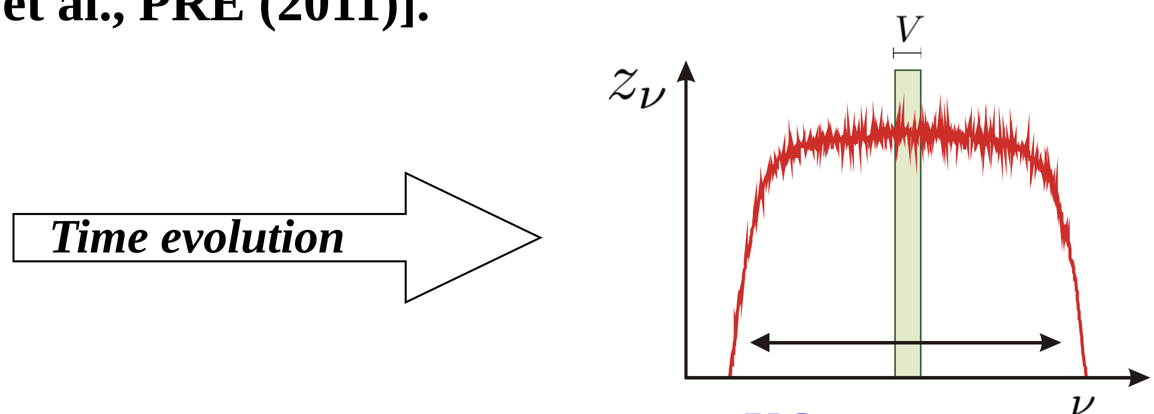
KG: Different spreading regimes



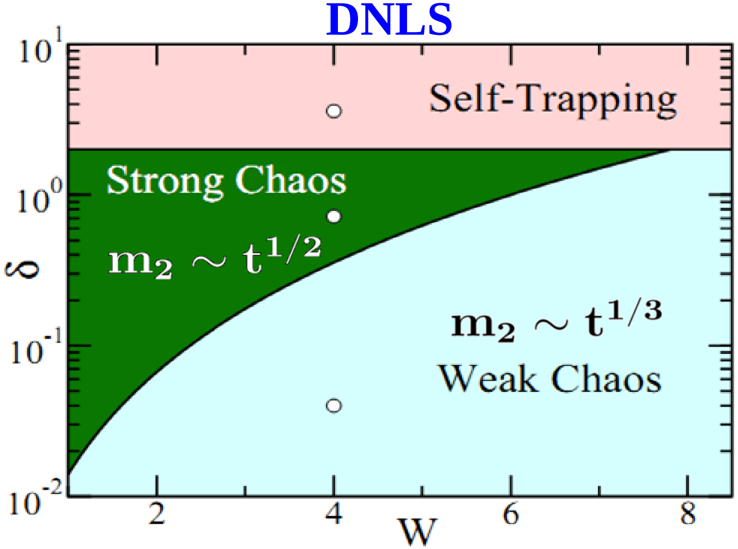
Crossover from strong to weak chaos

We consider **compact initial wave packets of width *L=V*** **[Laptyeva et al., EPL (2010) - Bodyfelt et al., PRE (2011)].**

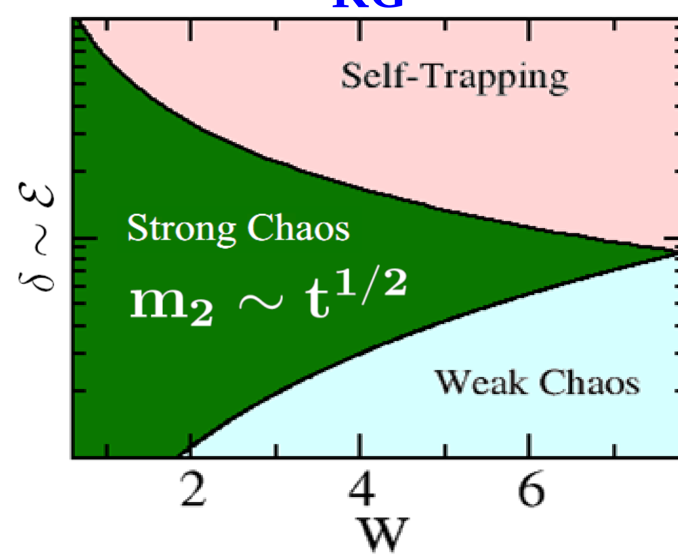




DNLS

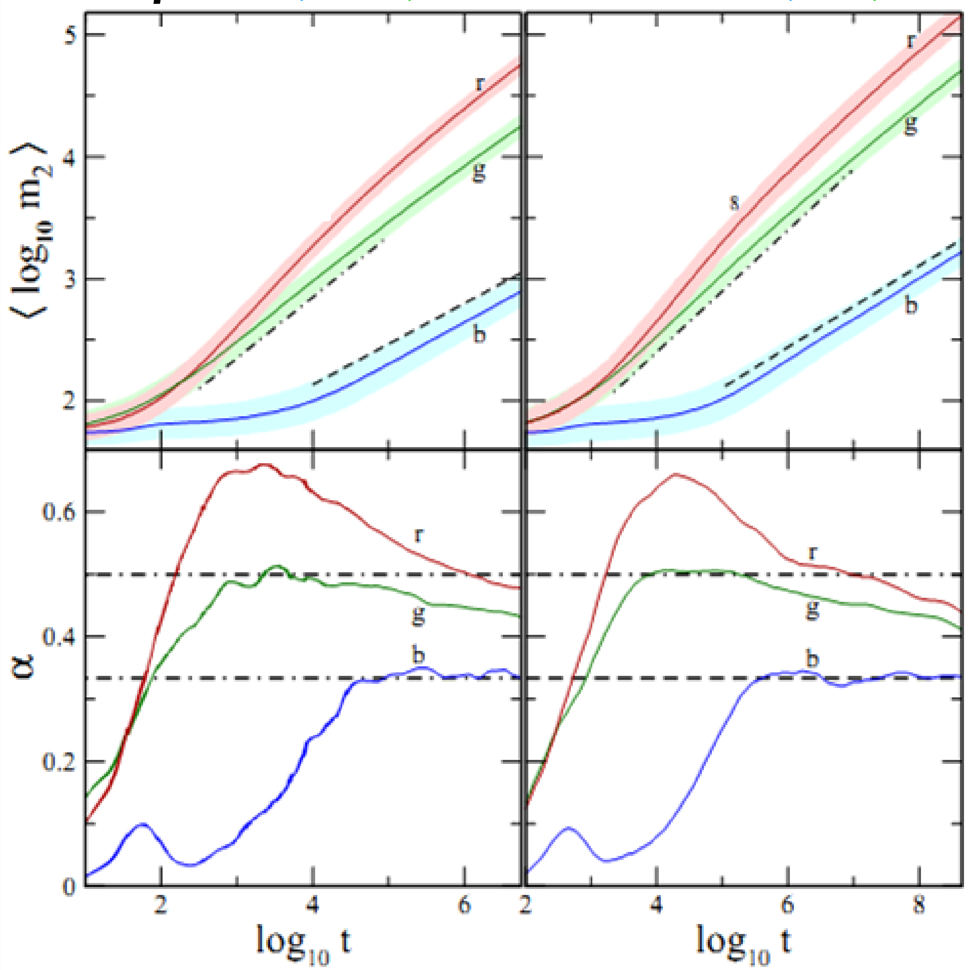


KG



Crossover from strong to weak chaos (block excitations)

DNLS β= 0.04, 0.72, 3.6 **KG E= 0.01, 0.2, 0.75**



α=1/2

α=1/3

W=4

Average over 1000 realizations!

$$\alpha(\log t) = \frac{d\langle \log m_2 \rangle}{d \log t}$$

Laptyeva et al., EPL (2010)

Bodyfelt et al., PRE (2011)

Lyapunov Exponents (LEs)

Roughly speaking, the Lyapunov exponents of a given orbit characterize the mean exponential rate of divergence of trajectories surrounding it.

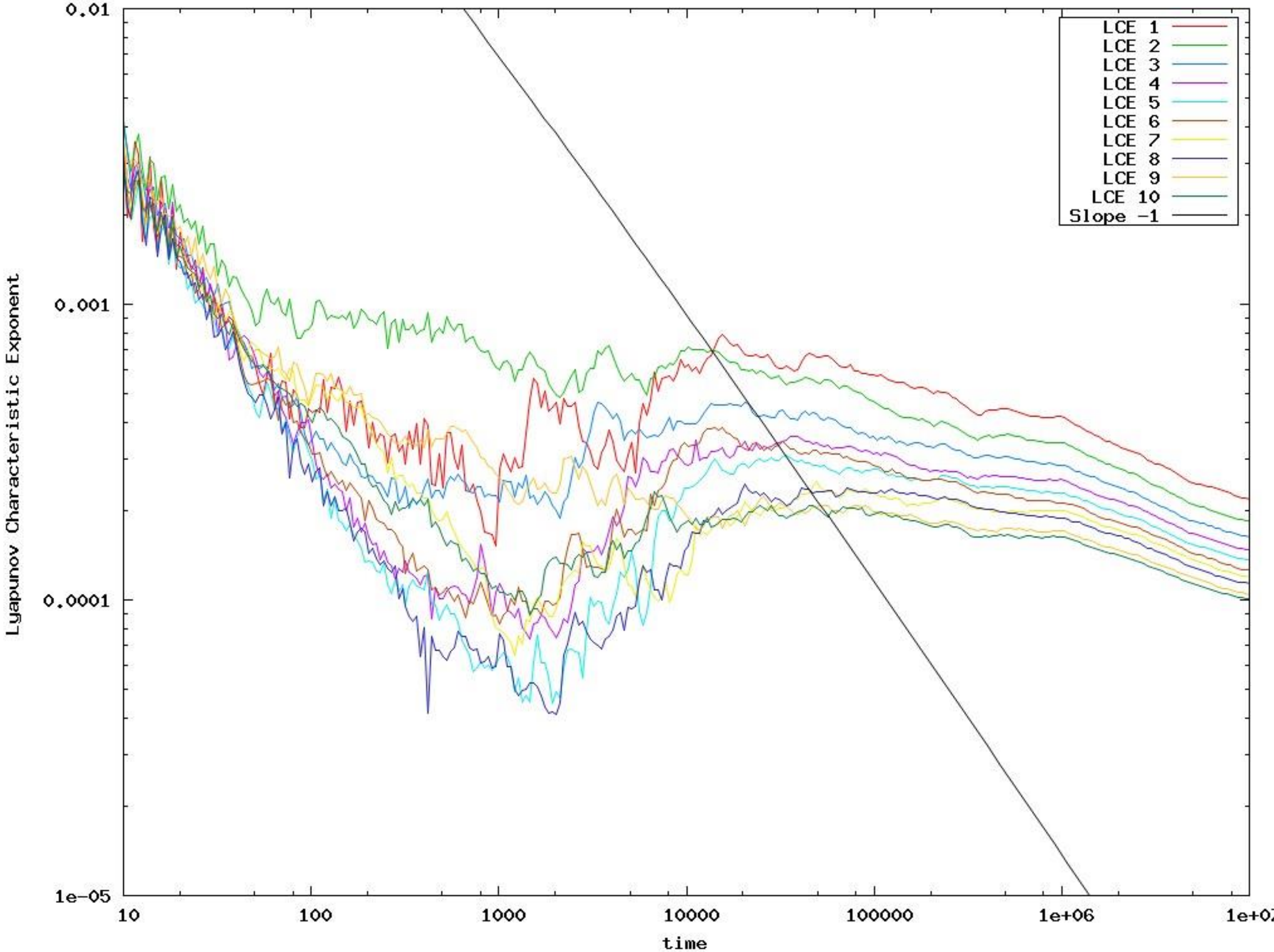
Consider an orbit in the 2N-dimensional phase space with initial condition x(0) and an initial deviation vector from it v(0). Then the mean exponential rate of divergence is:

$$\mathrm{mLCE} = \lambda_1 = \lim_{t \to \infty} \frac{1}{t} \ln \frac{\|\vec{v}(t)\|}{\|\vec{v}(0)\|}$$

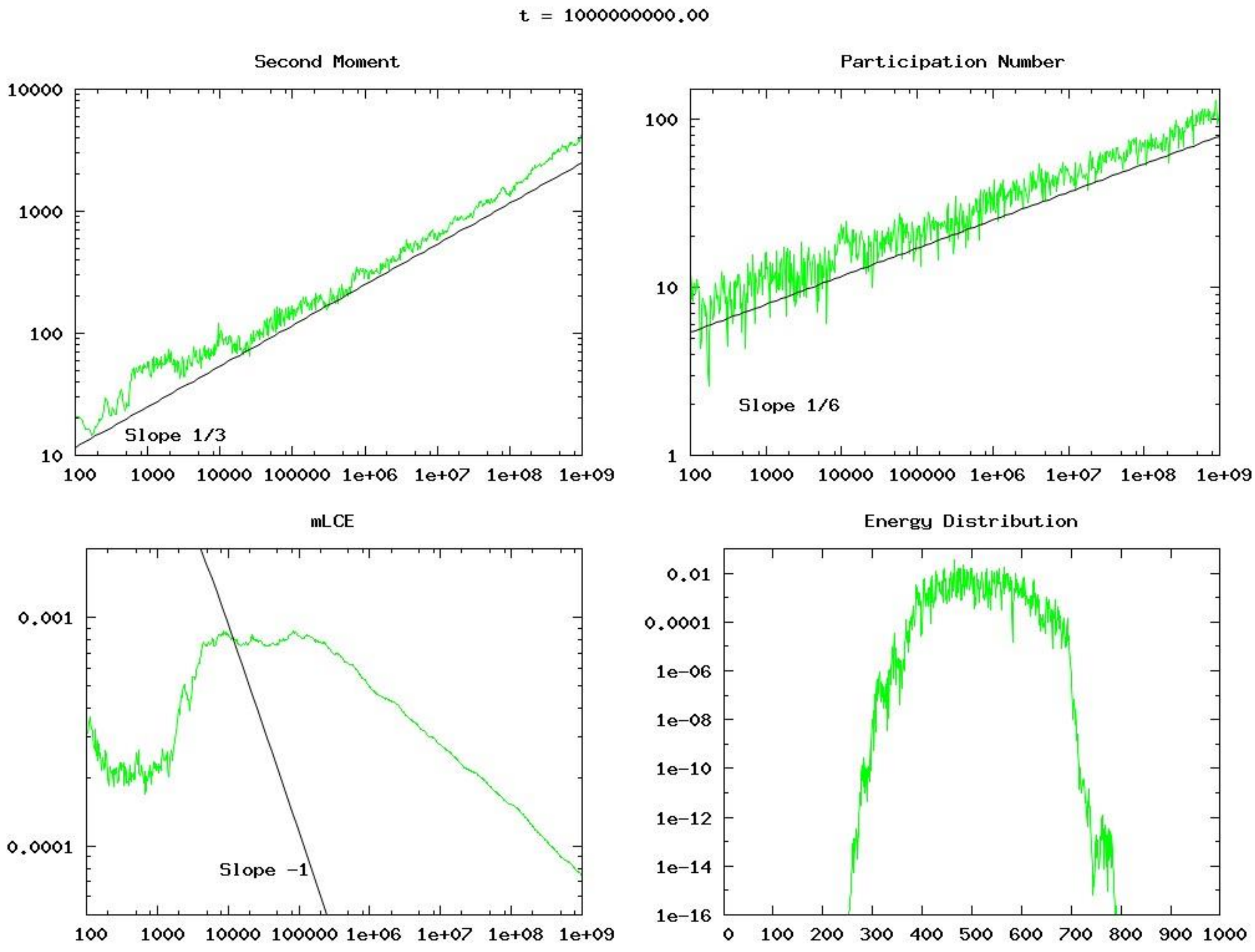
$\lambda_1 = 0 \rightarrow$ Regular motion $\propto (t^{-1})$

$\lambda_1 \neq 0 \rightarrow$ Chaotic motion

KG: LEs for single site excitations (E=0.4)

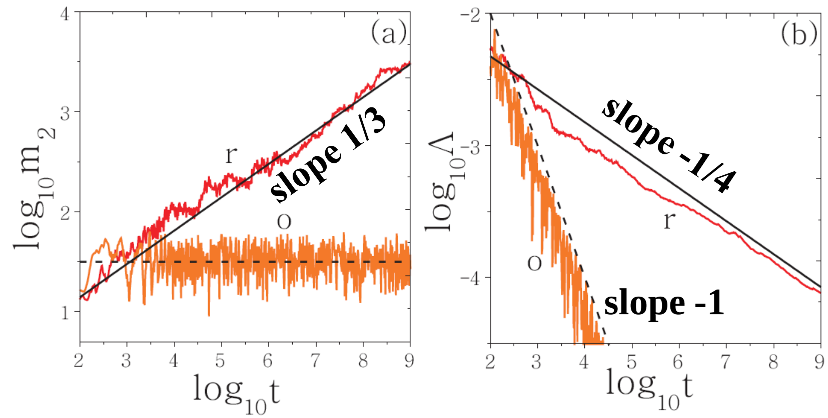


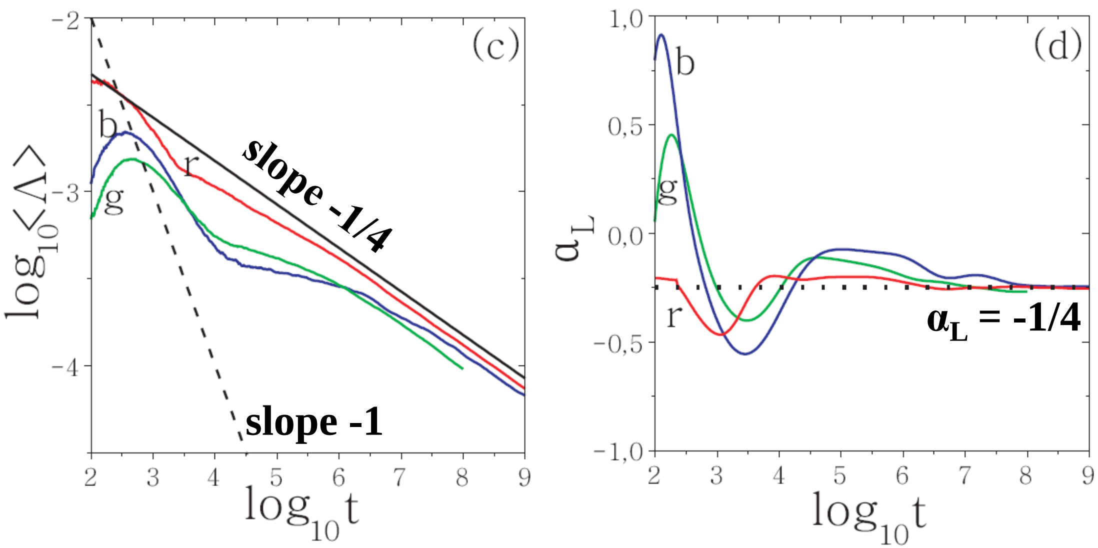
KG: Weak Chaos (E=0.4)



KG: Weak Chaos

Individual runs
Linear case
E=0.4, W=4





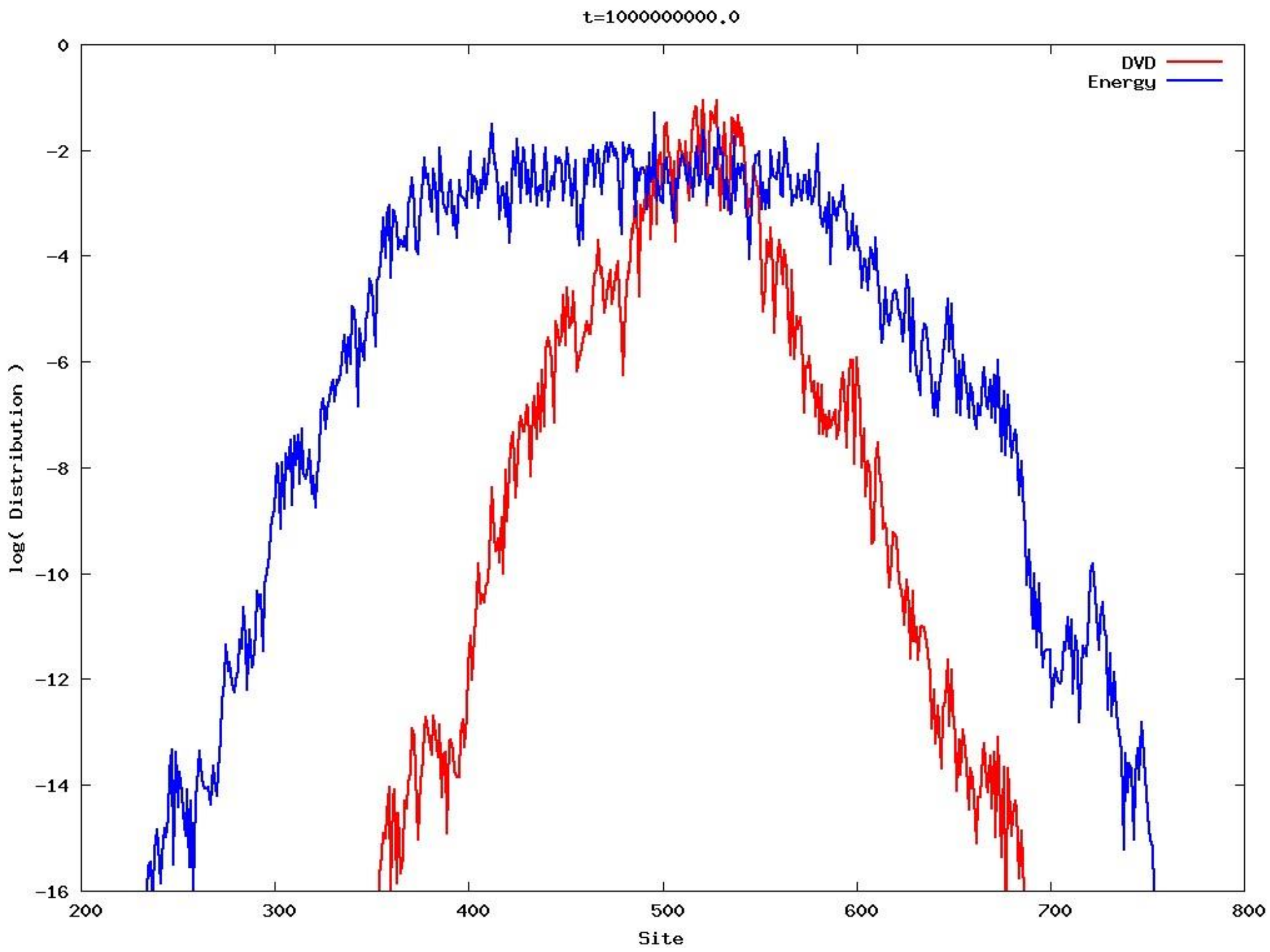
$$\alpha_L = \frac{d\left(\log\langle\Lambda\rangle\right)}{d\log t}$$

Average over 50 realizations

Single site excitation E=0.4, W=4
Block excitation (L=21 sites) E=0.21, W=4
Block excitation (L=37 sites) E=0.37, W=3

S. et al., PRL (2013)

Deviation Vector Distributions (DVDs)

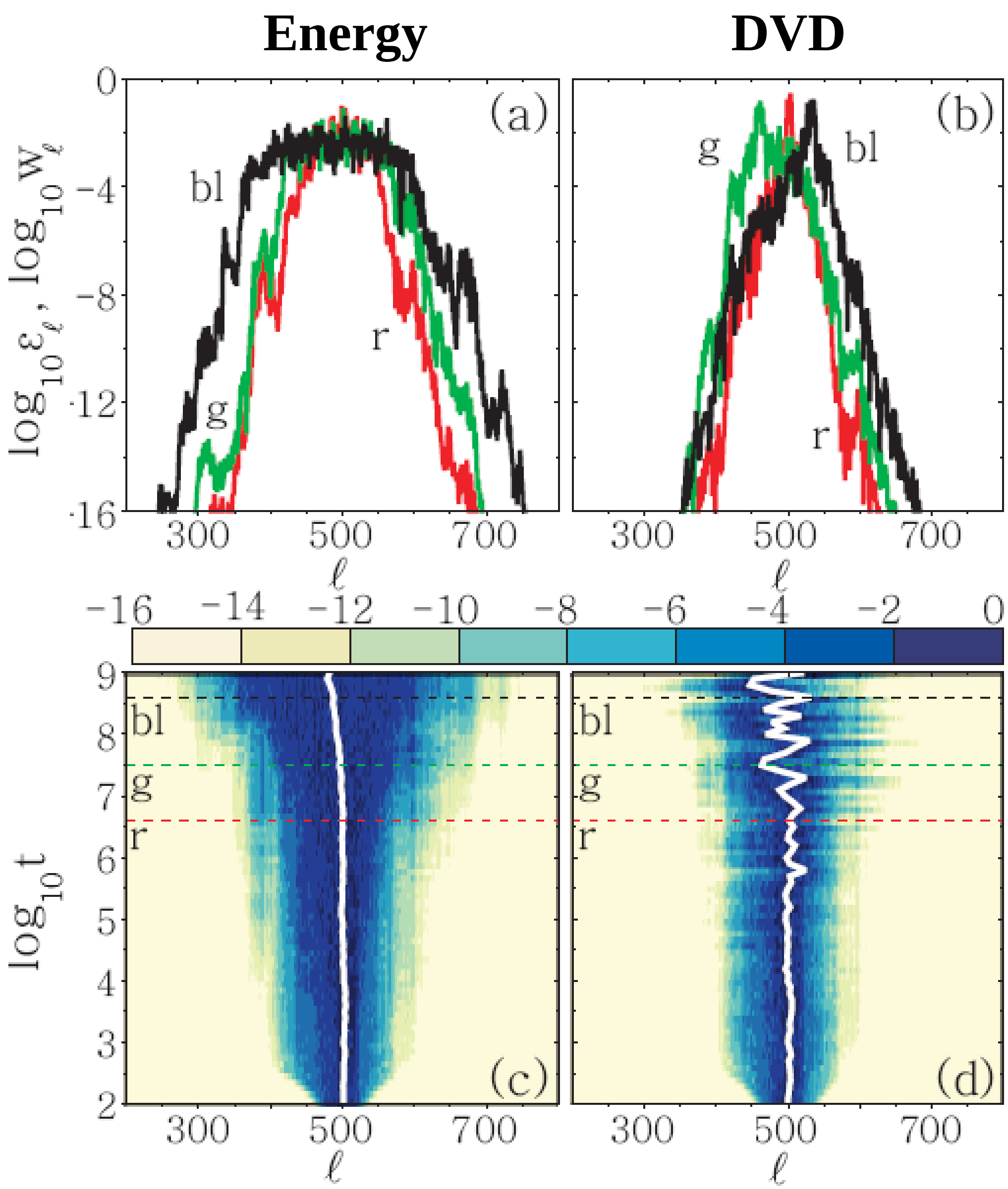


Deviation vector:

v(t)=(δu_1(t), δu_2(t),…, δu_N(t), δp_1(t), δp_2(t),…, δp_N(t))

DVD: $$w_l = \frac{\delta u_l^2 + \delta p_l^2}{\sum_l \left(\delta u_l^2 + \delta p_l^2\right)}$$

Deviation Vector Distributions (DVDs)



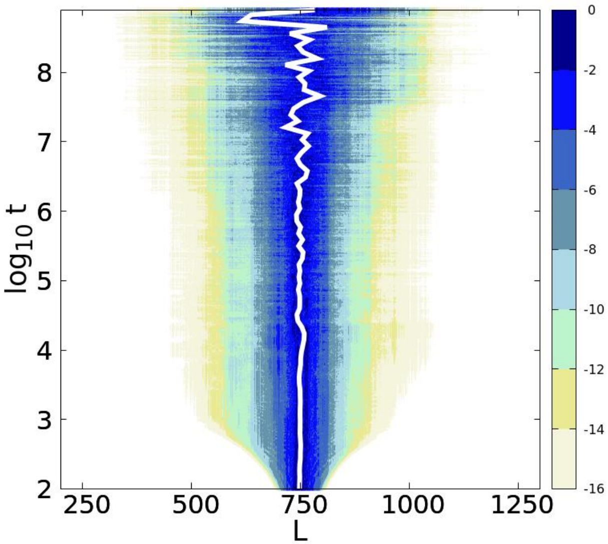
Individual run
E=0.4, W=4

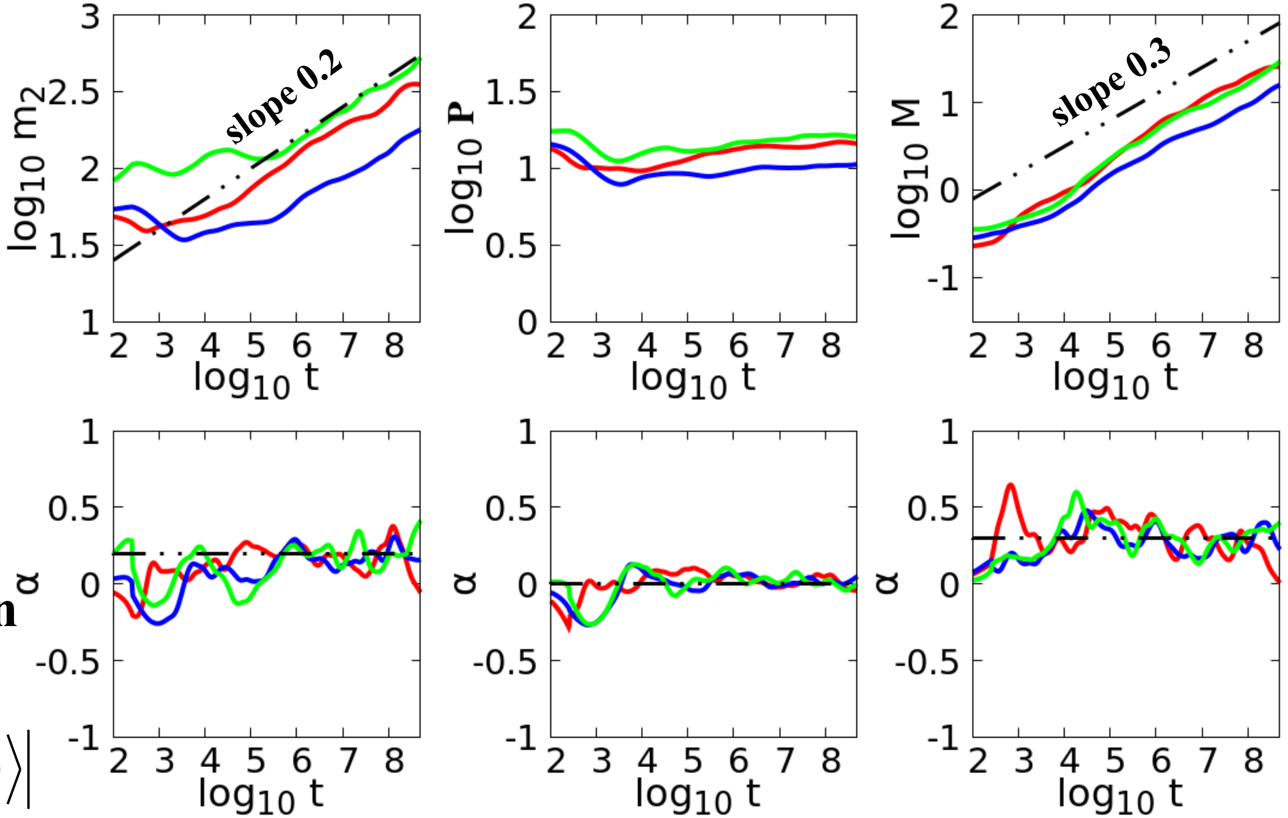
Chaotic hot spots meander through the system, supporting a homogeneity of chaos inside the wave packet.

DVDs – Weak chaos

Individual run, L=37, E=0.37, W=3

Single site excitation E=0.4, W=4
Block excitation (21 sites) E=0.21, W=4
Block excitation (37 sites) E=0.37, W=3

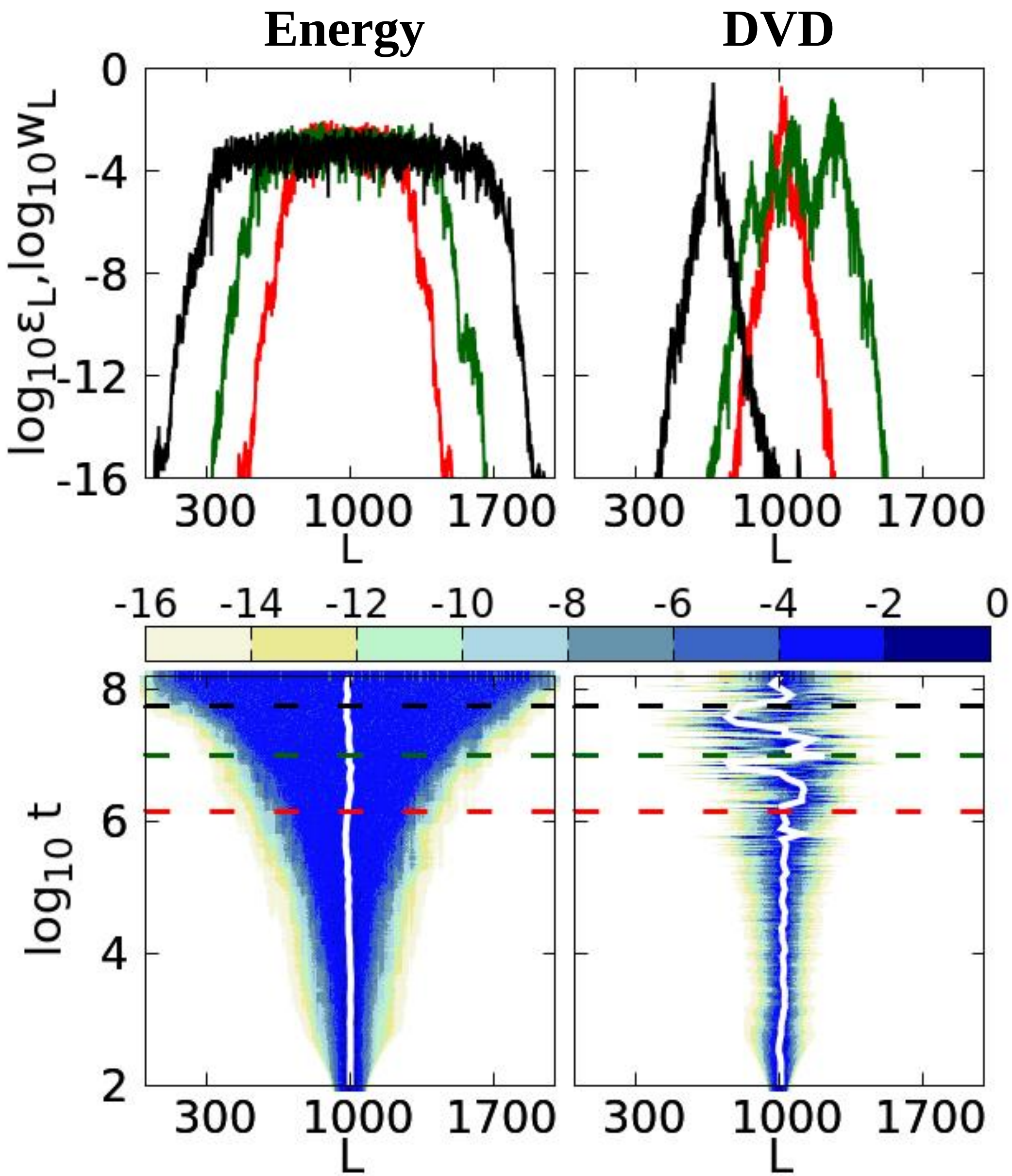




Maximum absolute deviation of DVD's mean position

$$M(t) = \max_{[t,t+\Delta t]} \left| \overline{w}(t) - \left\langle \overline{w}(t) \right\rangle \right|$$

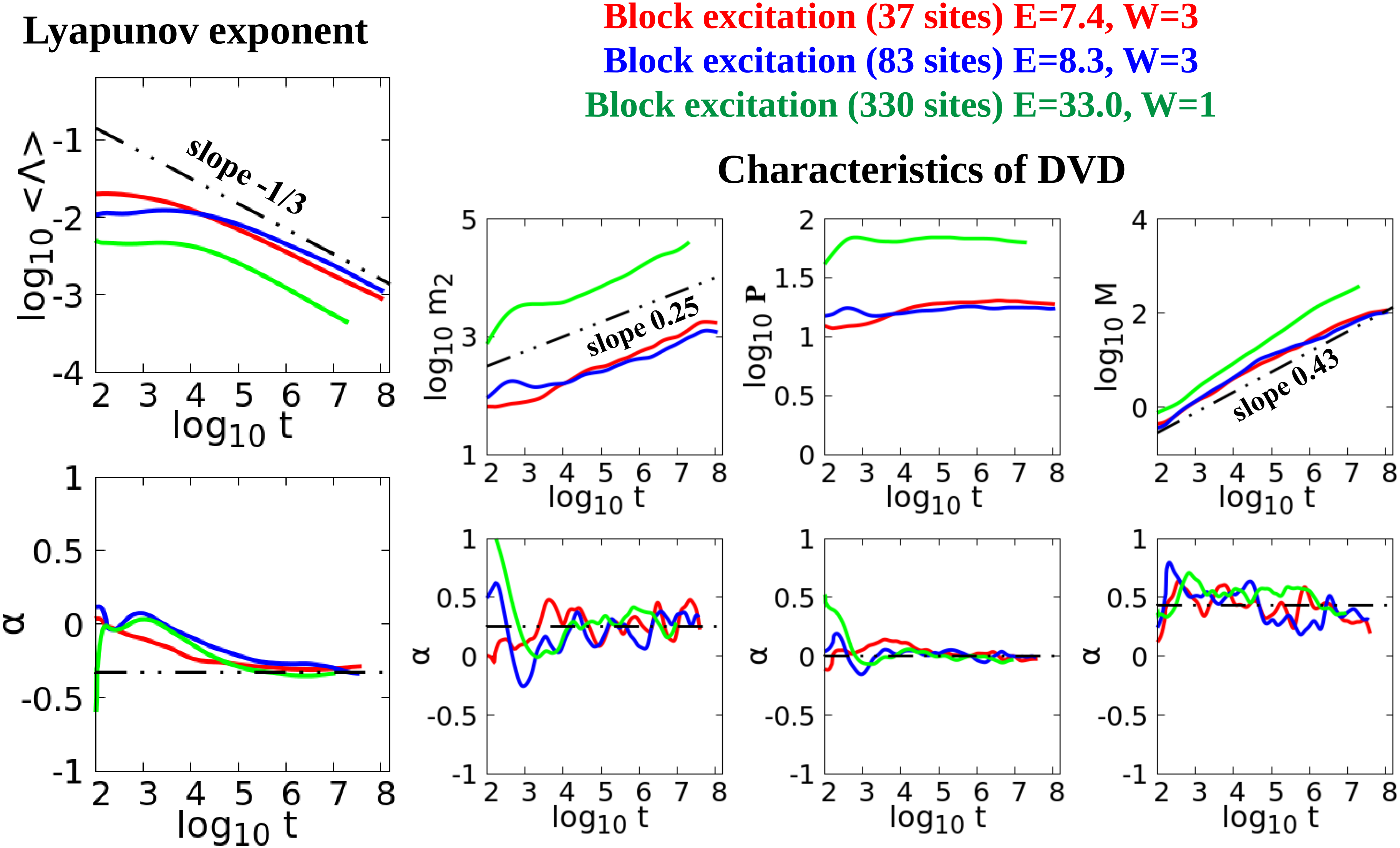
KG: Strong chaos



Individual run
L=83, E=8.3, W=3

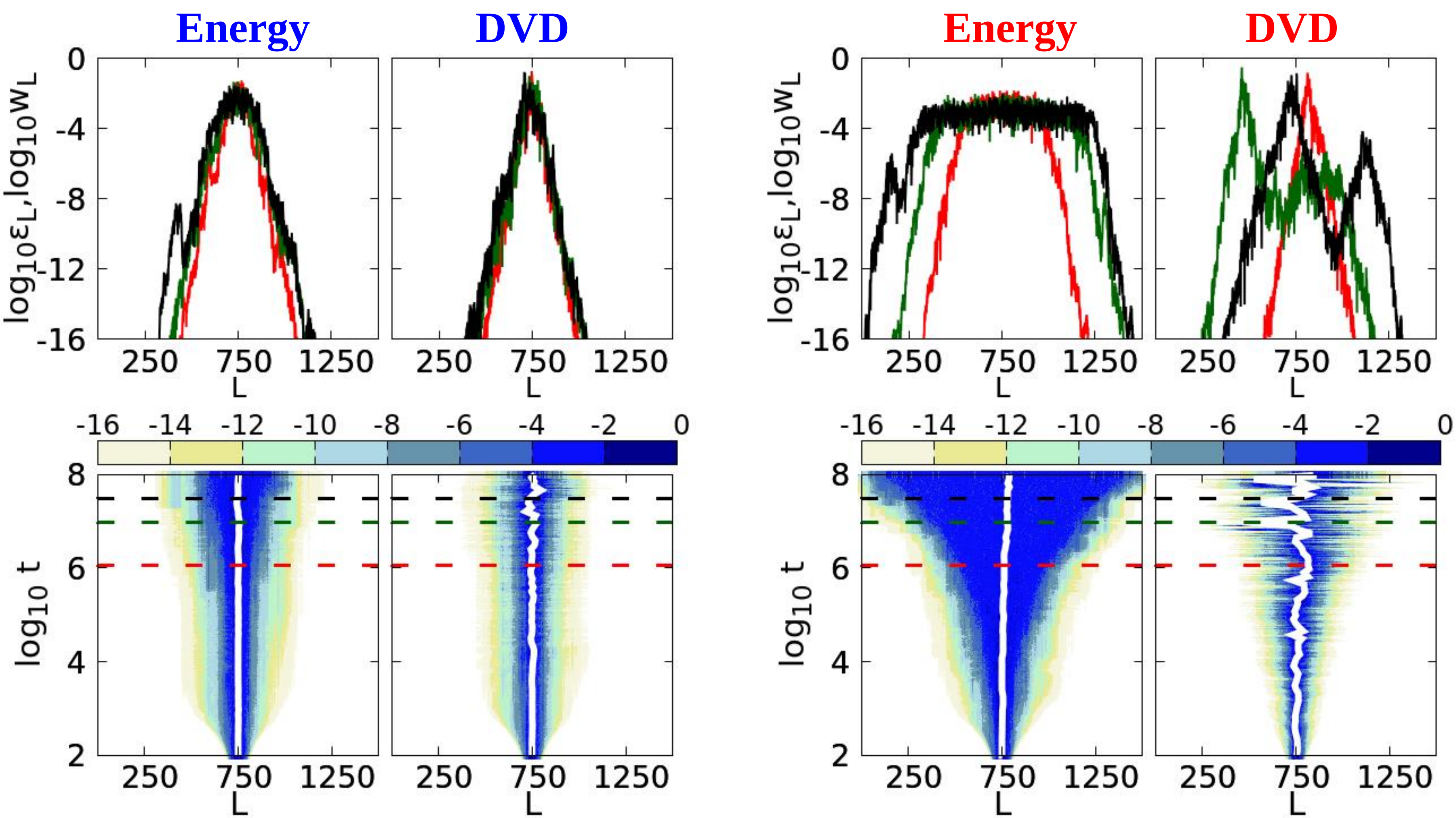
KG: Strong chaos

Block excitation (37 sites) E=7.4, W=3
Block excitation (83 sites) E=8.3, W=3
Block excitation (330 sites) E=33.0, W=1

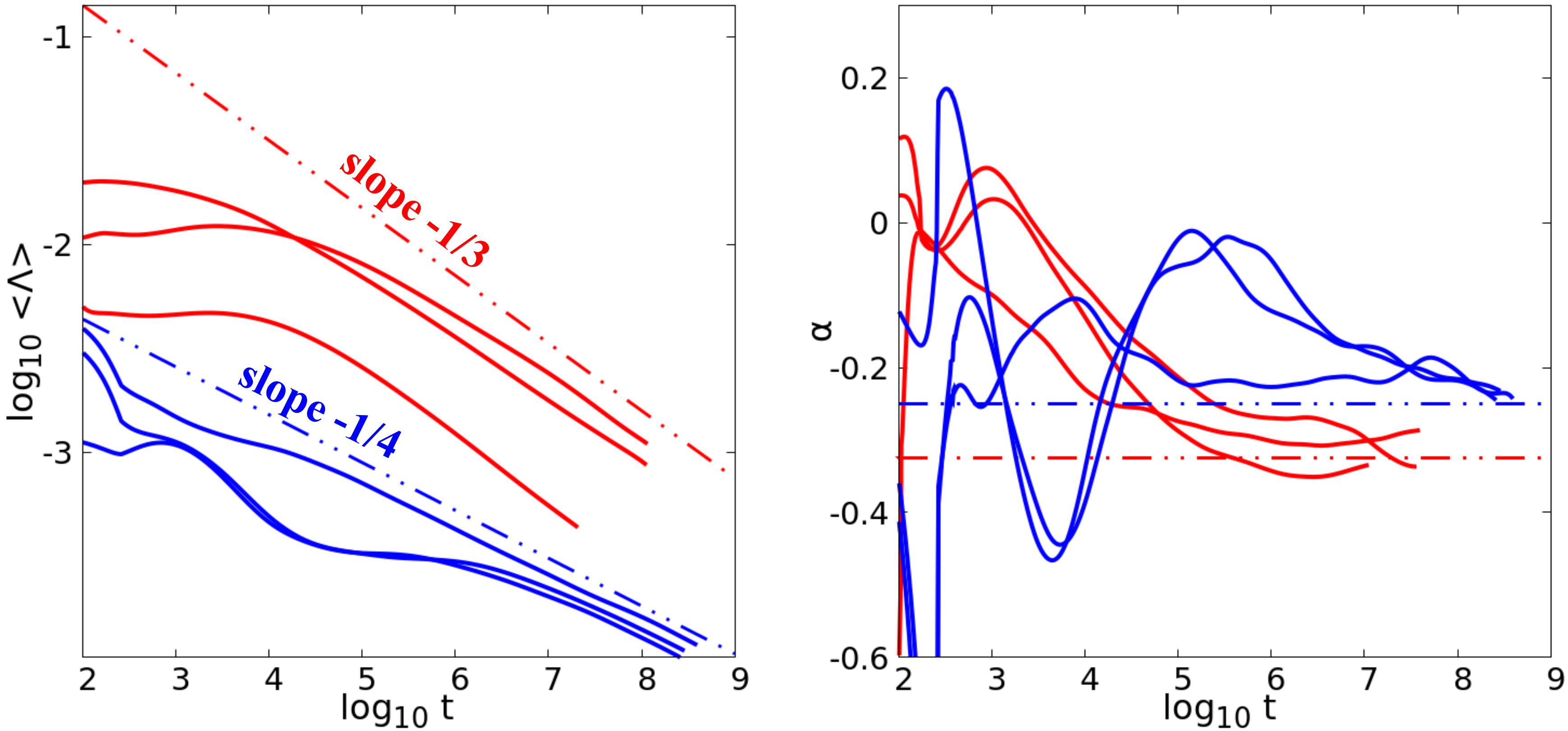


Weak and Strong chaos

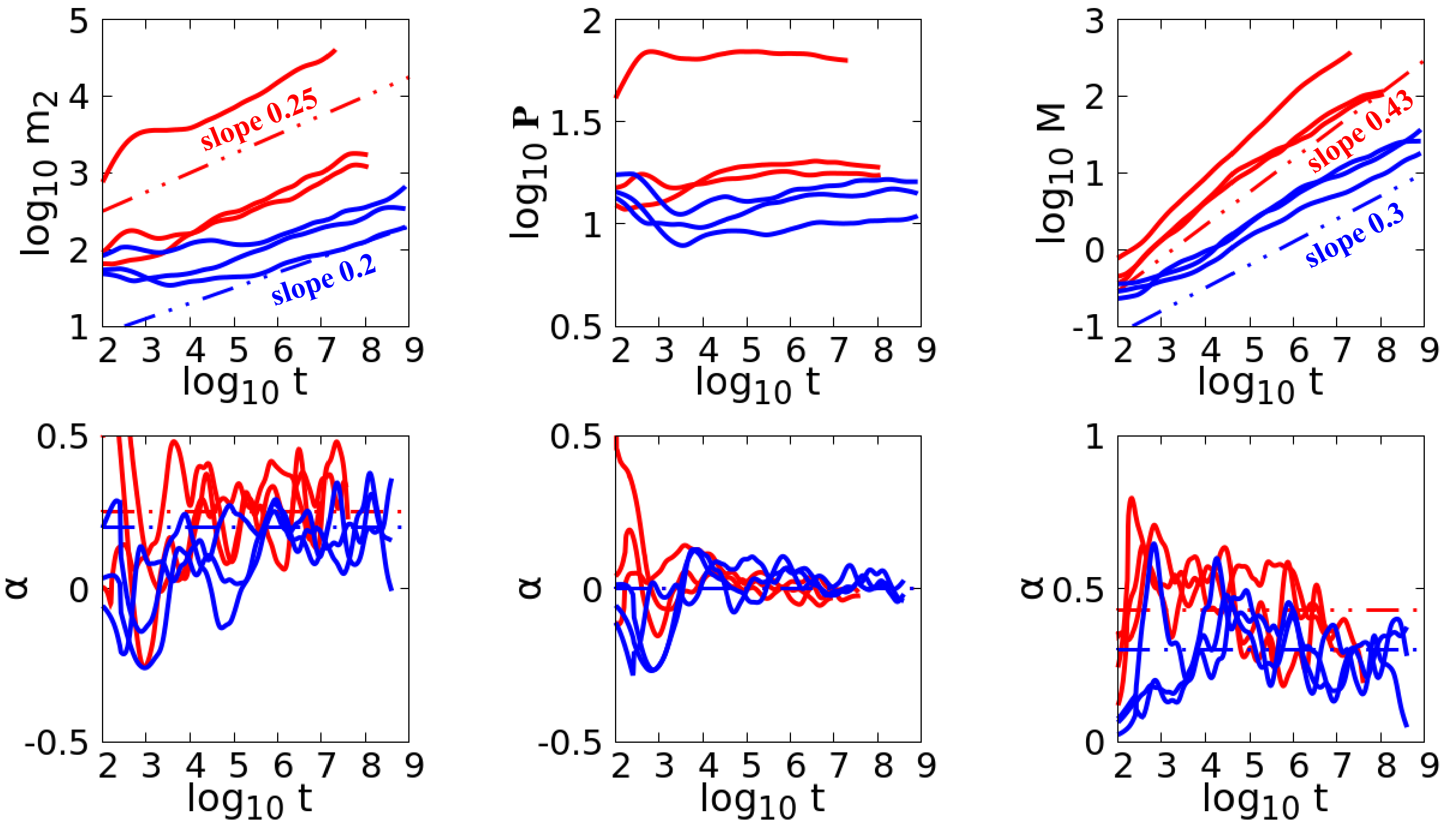
Same disordered realization, L=37, W=3, E=0.37 and E=7.4



Weak and Strong chaos: LEs



Weak and Strong chaos: DVDs



For both cases the DVD's participation number remains practically constant.

Autonomous Hamiltonian systems

Let us consider an N degree of freedom autonomous Hamiltonian systems of the form:

$$H(\vec{q},\vec{p}) = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + V(\vec{q})$$

As an example, we consider the Hénon-Heiles system:

$$H_2 = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(x^2 + y^2) + x^2 y - \frac{1}{3}y^3$$

Hamilton equations of motion:

$$\begin{cases} \dot{x} &= p_x \\ \dot{y} &= p_y \\ \dot{p}_x &= -x - 2xy \\ \dot{p}_y &= y^2 - x^2 - y \end{cases}$$

Variational equations:

$$\begin{cases} \dot{\delta x} &= \delta p_x \\ \dot{\delta y} &= \delta p_y \\ \dot{\delta p}_x &= -(1+2y)\delta x - 2x\delta y \\ \dot{\delta p}_y &= -2x\delta x + (-1+2y)\delta y \end{cases}$$

Symplectic Integrators (SIs)

Formally the solution of the Hamilton equations of motion can be written as:

$$\frac{d\vec{X}}{dt} = \left\{H, \vec{X}\right\} = L_H \vec{X} \Rightarrow \vec{X}(t) = \sum_{n\geq 0} \frac{t^n}{n!} L_H^n \vec{X} = e^{tL_H} \vec{X}$$

where $\vec{X}$ is the full coordinate vector and L_H the Poisson operator:

$$L_H f = \sum_{j=1}^{N} \left\{ \frac{\partial H}{\partial p_j} \frac{\partial f}{\partial q_j} - \frac{\partial H}{\partial q_j} \frac{\partial f}{\partial p_j} \right\}$$

If the Hamiltonian H can be split into two integrable parts as H=A+B, a symplectic scheme for integrating the equations of motion from time t to time t+τ consists of approximating the operator $e^{\tau L_H}$ by

$$e^{\tau L_H} = e^{\tau(L_A + L_B)} = \prod_{i=1}^{j} e^{c_i \tau L_A} e^{d_i \tau L_B} + O(\tau^{n+1})$$

for appropriate values of constants c_i, d_i. This is an integrator of order n.

<u>So the dynamics over an integration time step τ is described by a series of successive acts of Hamiltonians A and B.</u>

Symplectic Integrator SABA$_2$C

The operator $e^{\tau L_H}$ can be approximated by the symplectic integrator [Laskar & Robutel, Cel. Mech. Dyn. Astr. (2001)]:

$$SABA_2 = e^{c_1 \tau L_A} e^{d_1 \tau L_B} e^{c_2 \tau L_A} e^{d_1 \tau L_B} e^{c_1 \tau L_A}$$

with $c_1 = \frac{1}{2} - \frac{\sqrt{3}}{6},\ c_2 = \frac{\sqrt{3}}{3},\ d_1 = \frac{1}{2}.$

The integrator has only small positive steps and its error is of order 2.

In the case where *A* is quadratic in the momenta and *B* depends only on the positions the method can be improved by introducing a corrector *C*, having a small negative step:

$$C = e^{-\tau^3 \frac{c}{2} L_{\{\{A,B\},B\}}}$$

with $c = \frac{2 - \sqrt{3}}{24}.$

Thus the full integrator scheme becomes: *SABAC$_2$ = C (SABA$_2$) C* and its error is of order 4.

Tangent Map (TM) Method

Use symplectic integration schemes for the whole set of equations (S. & Gerlach, PRE (2010)

We apply the SABAC$_2$ integrator scheme to the Hénon-Heiles system by using the splitting:

$$A = \frac{1}{2}(p_x^2 + p_y^2), \qquad B = \frac{1}{2}(x^2 + y^2) + x^2 y - \frac{1}{3}y^3,$$

with a corrector term which corresponds to the Hamiltonian function:

$$C = \{\{A, B\}, B\} = (x + 2xy)^2 + \left(x^2 - y^2 + y\right)^2$$

We approximate the dynamics by the act of Hamiltonians A, B and C, which correspond to the symplectic maps:

$$e^{\tau L_A} : \begin{cases} x' &= x + p_x \tau \\ y' &= y + p_y \tau \\ p'_x &= p_x \\ p'_y &= p_y \end{cases},$$

$$e^{\tau L_B} : \begin{cases} x' &= x \\ y' &= y \\ p'_x &= p_x - x(1 + 2y)\tau \\ p'_y &= p_y + (y^2 - x^2 - y)\tau \end{cases},$$

$$e^{\tau L_C} : \begin{cases} x' &= x \\ y' &= y \\ p'_x &= p_x - 2x(1 + 2x^2 + 6y + 2y^2)\tau \\ p'_y &= p_y - 2(y - 3y^2 + 2y^3 + 3x^2 + 2x^2 y)\tau \end{cases}.$$

Tangent Map (TM) Method

Let $\vec{u} = (x, y, p_x, p_y, \delta x, \delta y, \delta p_x, \delta p_y)$

The system of the Hamilton's equations of motion and the variational equations is split into two integrable systems which correspond to Hamiltonians A and B.

$$\begin{aligned}
\dot{x} &= p_x \\
\dot{y} &= p_y \\
\dot{p}_x &= -x - 2xy \\
\dot{p}_y &= y^2 - x^2 - y \\
\dot{\delta x} &= \delta p_x \\
\dot{\delta y} &= \delta p_y \\
\dot{\delta p}_x &= -(1+2y)\delta x - 2x\delta y \\
\dot{\delta p}_y &= -2x\delta x + (-1+2y)\delta y
\end{aligned}$$

$$\xrightarrow{A(\vec{p})}\quad \left.\begin{aligned}
\dot{x} &= p_x \\
\dot{y} &= p_y \\
\dot{p}_x &= 0 \\
\dot{p}_y &= 0 \\
\dot{\delta x} &= \delta p_x \\
\dot{\delta y} &= \delta p_y \\
\dot{\delta p}_x &= 0 \\
\dot{\delta p}_y &= 0
\end{aligned}\right\} \Rightarrow \frac{d\vec{u}}{dt} = L_{AV}\vec{u} \quad \Rightarrow \quad e^{\tau L_{AV}} : \left\{\begin{aligned}
x' &= x + p_x\tau \\
y' &= y + p_y\tau \\
px' &= p_x \\
py' &= p_y \\
\delta x' &= \delta x + \delta p_x\tau \\
\delta y' &= \delta y + \delta p_y\tau \\
\delta p_x' &= \delta p_x \\
\delta p_y' &= \delta p_y
\end{aligned}\right.$$

$$\xrightarrow{B(\vec{q})}\quad \left.\begin{aligned}
\dot{x} &= 0 \\
\dot{y} &= 0 \\
\dot{p}_x &= -x - 2xy \\
\dot{p}_y &= y^2 - x^2 - y \\
\dot{\delta x} &= 0 \\
\dot{\delta y} &= 0 \\
\dot{\delta p}_x &= -(1+2y)\delta x - 2x\delta y \\
\dot{\delta p}_y &= -2x\delta x + (-1+2y)\delta y
\end{aligned}\right\} \Rightarrow \frac{d\vec{u}}{dt} = L_{BV}\vec{u} \quad \Rightarrow \quad e^{\tau L_{BV}} : \left\{\begin{aligned}
x' &= x \\
y' &= y \\
p_x' &= p_x - x(1+2y)\tau \\
p_y' &= p_y + (y^2 - x^2 - y)\tau \\
\delta x' &= \delta x \\
\delta y' &= \delta y \\
\delta p_x' &= \delta p_x - [(1+2y)\delta x + 2x\delta y]\tau \\
\delta p_y' &= \delta p_y + [-2x\delta x + (-1+2y)\delta y]\tau
\end{aligned}\right.$$

Tangent Map (TM) Method

Any symplectic integration scheme used for solving the Hamilton equations of motion, which involves the act of Hamiltonians A and B, can be extended in order to integrate simultaneously the variational equations [S. & Gerlach, PRE (2010) – Gerlach & S., Discr. Cont. Dyn. Sys. (2011) – Gerlach et al., IJBC (2012)].

$$e^{\tau L_A}: \begin{cases} x' &= x + p_x\tau \\ y' &= y + p_y\tau \\ p_x' &= p_x \\ p_y' &= p_y \end{cases}$$

$$e^{\tau L_{AV}}: \begin{cases} x' &= x + p_x\tau \\ y' &= y + p_y\tau \\ px' &= p_x \\ py' &= p_y \\ \delta x' &= \delta x + \delta p_x\tau \\ \delta y' &= \delta y + \delta p_y\tau \\ \delta p_x' &= \delta p_x \\ \delta p_y' &= \delta p_y \end{cases}$$

$$e^{\tau L_B}: \begin{cases} x' &= x \\ y' &= y \\ p_x' &= p_x - x(1+2y)\tau \\ p_y' &= p_y + (y^2 - x^2 - y)\tau \end{cases}$$

$$e^{\tau L_{BV}}: \begin{cases} x' &= x \\ y' &= y \\ p_x' &= p_x - x(1+2y)\tau \\ p_y' &= p_y + (y^2 - x^2 - y)\tau \\ \delta x' &= \delta x \\ \delta y' &= \delta y \\ \delta p_x' &= \delta p_x - [(1+2y)\delta x + 2x\delta y]\,\tau \\ \delta p_y' &= \delta p_y + [-2x\delta x + (-1+2y)\delta y]\,\tau \end{cases}$$

$$e^{\tau L_C}: \begin{cases} x' &= x \\ y' &= y \\ p_x' &= p_x - 2x(1+2x^2+6y+2y^2)\tau \\ p_y' &= p_y - 2(y - 3y^2 + 2y^3 + 3x^2 + 2x^2y)\tau \end{cases}$$

$$e^{\tau L_{CV}}: \begin{cases} x' &= x \\ y' &= y \\ p_x' &= p_x - 2x(1+2x^2+6y+2y^2)\tau \\ p_y' &= p_y - 2(y - 3y^2 + 2y^3 + 3x^2 + 2x^2y)\tau \\ \delta x' &= \delta x \\ \delta y' &= \delta y \\ \delta p_x' &= \delta p_x - 2\left[(1+6x^2+2y^2+6y)\delta x + \right. \\ & \quad \left. +2x(3+2y)\delta y\right]\tau \\ \delta p_y' &= \delta p_y - 2\left[2x(3+2y)\delta x + \right. \\ & \quad \left. +(1+2x^2+6y^2-6y)\delta y\right]\tau \end{cases}$$

The KG model

We apply the SABAC$_2$ integrator scheme to the KG Hamiltonian by using the splitting:

$$H_K = \sum_{l=1}^{N}\left(\underbrace{\frac{p_l^2}{2}}_{\mathbf{A}} + \underbrace{\frac{\tilde{\varepsilon}_l}{2}u_l^2 + \frac{1}{4}u_l^4 + \frac{1}{2W}\left(u_{l+1} - u_l\right)^2}_{\mathbf{B}} \right)$$

$$e^{\tau L_A}: \begin{cases} u_l' = p_l\tau + u_l \\ p_l' = p_l, \end{cases}$$

$$e^{\tau L_B}: \begin{cases} u_l' = u_l \\ p_l' = \left[-u_l(\tilde{\epsilon}_l + u_l^2) + \frac{1}{W}(u_{l-1} + u_{l+1} - 2u_l) \right]\tau + p_l, \end{cases}$$

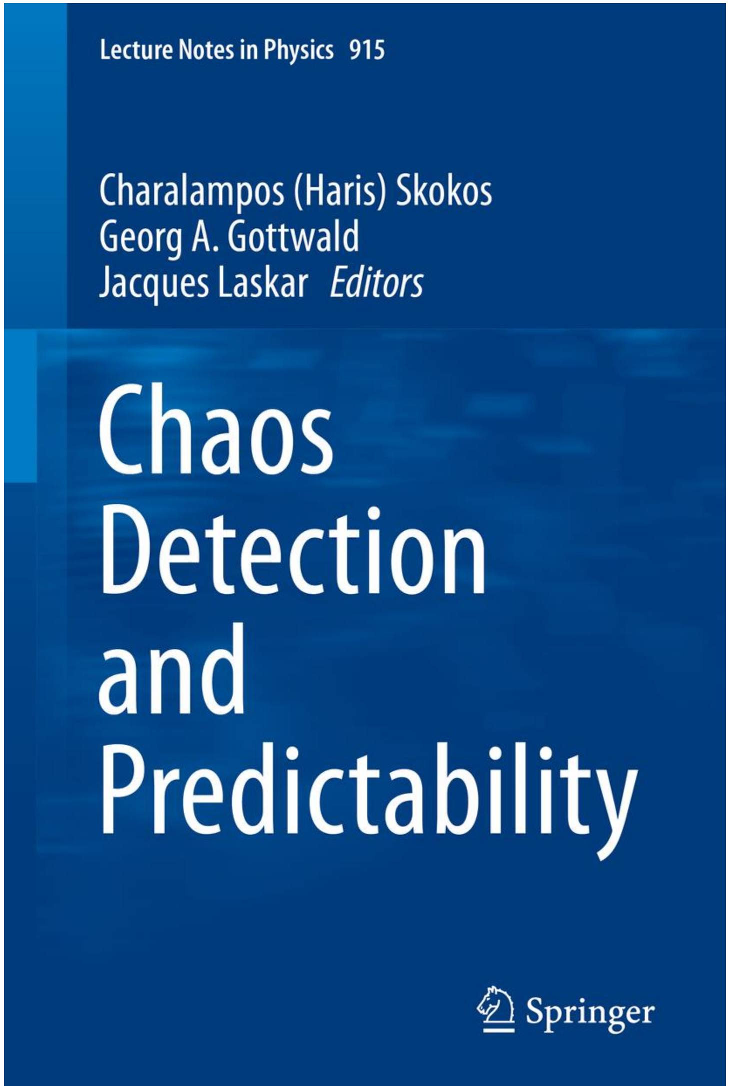
with a corrector term which corresponds to the Hamiltonian function:

$$\mathbf{C} = \left\{\left\{A, B\right\}, B\right\} = \sum_{l=1}^{N}\left[u_l(\tilde{\varepsilon}_l + u_l^2) - \frac{1}{W}(u_{l-1} + u_{l+1} - 2u_l) \right]^2 .$$

Summary

- **We presented three different dynamical behaviors for wave packet spreading in 1d nonlinear disordered lattices (KG and DNLS models):**
  - ✓ **Weak Chaos Regime: $\delta<d$, $m_2 \sim t^{1/3}$**
  - ✓ **Intermediate Strong Chaos Regime: $d<\delta<\Delta$, $m_2 \sim t^{1/2} \longrightarrow m_2 \sim t^{1/3}$**
  - ✓ **Selftrapping Regime: $\delta>\Delta$**
- **KG model**
  - ✓ **Lyapunov exponent computations show that:**
    - **Chaos not only exists, but also persists.**
    - **Slowing down of chaos does not cross over to regular dynamics.**
  - ✓ **mLEs and DVDs show different behaviors for the weak and the strong chaos regimes.**
  - ✓ **Chaotic hot spots meander through the system, supporting a homogeneity of chaos inside the wave packet.**
- **The behavior of DVDs can provide important information about the chaotic behavior of a dynamical system.**

A …shameless promotion



2016, Lect. Notes Phys., 915, Springer

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